The effect of the rotation of the central body on the orbit of a particle

1974 J. Phys. A: Math. Nucl. Gen. 7 216

(http://iopscience.iop.org/0301-0015/7/2/008)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 2, 1974. Printed in Great Britain. © 1974

The effect of the rotation of the central body on the orbit of a particle

Brendan Breen†

Mathematics Department, Imperial College of Science and Technology, London SW7 2RH, UK

Received 10 August 1973

Abstract. The general postnewtonian metric expansion of Nordtvedt is modified in such a way that the effects of the rotation of the central body on the orbit of a test particle can be examined. This enables formulae to be found for the additional advance of the perihelion and rate of precession of the normal of the orbit about the axis of rotation in general relativity and in the Brans–Dicke and Nordtvedt scalar–tensor theories. In the case of general relativity the rate of precession is found to be half the value previously given by Lense and Thirring.

1. Introduction

In a previous paper (Breen 1973, to be referred to as I) some of the consequences of the general postnewtonian metric expansion of Nordtvedt were considered. For n moving sources this metric is:

$$g_{00} = 1-2\alpha\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}+2\beta\left(\sum_i \frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)^2+2\alpha'\sum_i\sum_{j\neq i}\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\cdot\frac{m_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}$$
$$+\frac{\chi}{c^2}\sum_i\frac{m_i(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{a}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}-\frac{4\alpha''}{c^2}\sum_i\frac{m_i v_i^2}{|\boldsymbol{r}-\boldsymbol{r}_i|}+\frac{\alpha'''}{c^2}\sum_i\frac{m_i[(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{v}_i]^2}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}, \qquad (1a)$$

$$g_{0k} = \frac{4\Delta}{c}\sum_i\frac{m_i(\boldsymbol{v}_i)_k}{|\boldsymbol{r}-\boldsymbol{r}_i|}, \qquad k = 1,2,3, \qquad (1b)$$

$$g_{kl} = -\left(1+2\gamma\sum_i\frac{m_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}\right)\delta_{kl}, \qquad k,l = 1,2,3, \qquad (1c)$$

where $m_i, \boldsymbol{r}_i, \boldsymbol{v}_i$ and $\boldsymbol{a}_i$ are respectively the geometrized mass, position, velocity and acceleration of the ith source ($i = 1,\ldots,n$). α, β, γ, χ, α', α'', α''' and Δ are dimensionless constants taking specific values in particular gravitational theories (see I for the values of these parameters in general relativity, the Brans–Dicke theory and the Nordtvedt scalar–tensor theory).

In the present paper the metric of equation (1) is modified so that the effect of the rotation of the central body on the orbit of a test particle can be considered. These effects were first examined in general relativity by Lense and Thirring (1918) who found that (i) there is an additional advance of the perihelion and (ii) the normal to the plane of the

† Now at Grimsby College of Technology, Nuns Corner, Grimsby, Lincolnshire, UK.

orbit precesses about the axis of rotation of the central body. These two effects will be examined here for the general metric and this will enable the values to be found for the three theories considered in I. In the case of general relativity it is found that the additional advance of the perihelion agrees with the well known result, but the rate of precession of the normal of the orbit is half the value found by Lense and Thirring.

2. Metric and equations of motion

The metric of equation (1) will now be integrated in such a way as to produce the exterior metric of a uniform sphere rotating with constant angular velocity Ω. (The rotation will be assumed to be slow, ie the velocity at the surface of the sphere is small compared with the velocity of light.) Let the n sources in equation (1) be moving so that they form a (discontinuous) body rotating with angular velocity Ω. Choose coordinates (x, y, z) so that (initially) the orbit of the particle P is in the (x, y) plane and the axis of rotation is in the (y, z) plane making an angle λ with the z axis (see figure 1). Then the number of sources can be allowed to tend to infinity so as to produce in the limit a continuous sphere of uniform density. This is a justifiable procedure since as the orbiting particle is necessarily outside the central body no singularities can occur. The only apparent exception is the α' term of g_{00}, but this can be dealt with by the usual method of excluding a sphere of radius ϵ around the singular point, evaluating the integral, and then letting $\epsilon \to 0$.

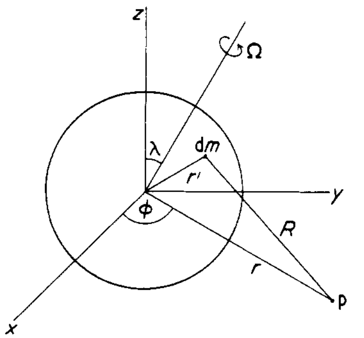

Figure 1. Configuration of coordinates for the central sphere and the orbiting particle P.

The new forms of the α and β terms of g_{00} and the γ term of g_{kl} can be written down immediately as these only involve the expression that would occur in newtonian theory for finding the potential of a point outside a sphere. The χ, α', α'' and α''' terms of g_{00} will be ignored here as the α' term makes no contribution to the effects to be considered, and the χ, α'' and α''' terms only make contributions of the order of Ω^2. Thus the integrated forms of equations ($1a$) and ($1c$) are

$$g_{00} = 1 - 2\alpha\frac{m}{r} + 2\beta\frac{m^2}{r^2} \tag{2a}$$

and

$$g_{kl} = -\left(1 + 2\gamma\frac{m}{r}\right)\delta_{kl} \tag{2b}$$

where $m = GM/c^2$ is the geometrized mass of the sphere. The integrated form of equation (1b) is

$$g_{0k} = \frac{4\Delta}{c}\int_{\text{over sphere}} \frac{\mathrm{d}m}{R}\left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_k$$

where

$$\mathrm{d}m = \frac{G\rho}{c^2}\,\mathrm{d}V' = \frac{G\rho}{c^2}r'^2\,\mathrm{d}r'\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi',$$

ρ is the constant density of the sphere, and $R = |\boldsymbol{r}-\boldsymbol{r}'|$ (see figure 1). Now

$$\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} = \boldsymbol{\Omega}\times\boldsymbol{r}'$$

where

$$\boldsymbol{\Omega} = (0, \Omega\sin\lambda, \Omega\cos\lambda)$$

and

$$\boldsymbol{r}' = (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta'),$$

so that

$$\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} = (\Omega\sin\lambda\, r'\cos\theta' - \Omega\cos\lambda\, r'\sin\theta'\sin\phi', \Omega\cos\lambda\, r'\sin\theta'\cos\phi', -\Omega\sin\lambda\, r'\sin\theta'\cos\phi').$$

Since

$$\boldsymbol{r} = (r\cos\phi, r\sin\phi, 0)$$

it follows that

$$R = |\boldsymbol{r}-\boldsymbol{r}'| = [r^2 + r'^2 - 2rr'(\sin\theta'\cos\phi'\cos\phi + \sin\theta'\sin\phi'\sin\phi)]^{1/2}$$

and hence,

$$\frac{1}{R} = \frac{1}{r}\left(1 + \frac{r'}{r}(\sin\theta'\cos\phi'\cos\phi + \sin\theta'\sin\phi'\sin\phi) - \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{2}\frac{r'^2}{r^2}(\sin\theta'\cos\phi'\cos\phi + \sin\theta'\sin\phi'\sin\phi)^2 + \ldots\right)$$

where it is assumed that terms of the order $(a/r)^3$, where a is the radius of the sphere, are negligible. Straightforward, but laborious, calculations then yield:

$$\begin{aligned} g_{01} &= -\frac{4\Delta ma^2}{5c}\frac{y}{r^3}\Omega\cos\lambda, \\ g_{02} &= +\frac{4\Delta ma^2}{5c}\frac{x}{r^3}\Omega\cos\lambda, \\ g_{03} &= -\frac{4\Delta ma^2}{5c}\frac{x}{r^3}\Omega\sin\lambda. \end{aligned} \qquad (2c)$$

The equations of motion of the orbiting particle are obtained by substituting the above equations (2) into equation (8), via equation (9), of I. The result is the following three scalar equations:

$$\ddot{x} = -\alpha mc^2\frac{x}{r^3}+(2\beta+2\alpha\gamma)m^2c^2\frac{x}{r^4}-\gamma mv^2\frac{x}{r^3}+(2\alpha+2\gamma)m(\boldsymbol{r}\cdot\boldsymbol{v})\frac{\dot{x}}{r^3}-\frac{4\Delta ma^2\Omega\cos\lambda}{5r^3}$$
$$\times\left(2\dot{y}-3(\boldsymbol{r}\cdot\boldsymbol{v})\frac{y}{r^2}\right)+\frac{12\Delta ma^2\Omega\cos\lambda}{5r^5}x(x\dot{y}-y\dot{x})+\mathrm{O}(c^{-4}), \qquad (3)$$

$$\ddot{y} = -\alpha mc^2\frac{y}{r^3}+(2\beta+2\alpha\gamma)m^2c^2\frac{y}{r^4}-\gamma mv^2\frac{y}{r^3}+(2\alpha+2\gamma)m(\boldsymbol{r}\cdot\boldsymbol{v})\frac{\dot{y}}{r^3}+\frac{4\Delta ma^2\Omega\cos\lambda}{5r^3}$$
$$\times\left(2\dot{x}-3(\boldsymbol{r}\cdot\boldsymbol{v})\frac{x}{r^2}\right)+\frac{12\Delta ma^2\Omega\cos\lambda}{5r^5}y(x\dot{y}-y\dot{x})+\mathrm{O}(c^{-4}), \qquad (4)$$

$$\ddot{z} = -\alpha mc^2\frac{z}{r^3}-\frac{4\Delta ma^2\Omega\sin\lambda}{5r^3}\left(\dot{x}-3(\boldsymbol{r}\cdot\boldsymbol{v})\frac{x}{r^2}\right)+\mathrm{O}(c^{-4}). \qquad (5)$$

In deriving equations (3), (4) and (5) use has been made of the fact that the orbit is a plane in newtonian theory, ie z, $\dot{z}$ and $\ddot{z}$ are all of order c^{-2}. Also, as will be seen in § 4, λ is a constant to order c^{-2}.

It may be noted that these equations can also be obtained directly from equation (10) of I, by letting the number of sources tend to infinity and integrating in the manner outlined above. But the calculations involved are even more laborious than those outlined here.

3. Advance of perihelion

The addition to the perihelion advance caused by the slow rotation of the central body is obtained from equations (3) and (4). The method used here is similar to that of § 5 of I.

On changing to polar coordinates $x = r\cos\phi$, $y = r\sin\phi$ (in the (x, y) plane of figure 1), equations (3) and (4) become

$$\ddot{r}-r\dot{\phi}^2 = -\alpha\frac{mc^2}{r^2}+(2\beta+2\alpha\gamma)\frac{m^2c^2}{r^3}-\gamma\frac{m}{r^2}(\dot{r}^2+r^2\dot{\phi}^2)+2(\alpha+\gamma)\frac{m\dot{r}^2}{r^2}$$
$$+\frac{4\Delta ma^2\Omega\cos\lambda}{5r^2}\dot{\phi}+\mathrm{O}(c^{-4}) \qquad (6)$$

and

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) = 2(\alpha+\gamma)\frac{m}{r}\dot{r}\dot{\phi}-\frac{4\Delta ma^2\Omega\cos\lambda}{5r^3}\dot{r}+\mathrm{O}(c^{-4}). \qquad (7)$$

Integrating equation (7) in the newtonian approximation gives $r^2\dot{\phi} = h+\mathrm{O}(c^{-2})$, where h is a constant. Using this result, equation (7) can be integrated in the postnewtonian approximation as

$$r^2\dot{\phi} = h+\frac{m}{r}\left(\frac{4\Delta a^2\Omega\cos\lambda}{5}-2(\alpha+\gamma)h\right)+\mathrm{O}(c^{-4}). \qquad (8)$$

On writing u for $1/r$, using equation (8) and the fact that

$$\frac{d^2u}{d\phi^2}+u = \alpha\frac{mc^2}{h^2}+O(c^{-2}), \tag{9}$$

equation (6) can be written as

$$\frac{d^2u}{d\phi^2}+u = \alpha\frac{mc^2}{h^2}+(4\alpha^2+2\alpha\gamma-2\beta)\frac{m^2c^2}{h^2}u+\gamma m\left[u^2+\left(\frac{du}{d\phi}\right)^2\right]$$
$$-\frac{4\Delta ma^2\Omega\cos\lambda}{5h}\left[u^2+\left(\frac{du}{d\phi}\right)^2\right]-\frac{8\Delta\alpha m^2a^2c^2\Omega\cos\lambda\, u}{5h^3}+O(c^{-4}). \tag{10}$$

From equation (9) it follows that

$$u = \alpha\frac{mc^2}{h^2}(1+\epsilon\cos(\phi-\phi_0))+O(c^{-2}), \tag{11}$$

where ϵ is the eccentricity of the orbit and ϕ_0 is a constant. On substituting equation (11) into the right-hand side of equation (10) it becomes

$$\frac{d^2u}{d\phi^2}+u = \alpha\frac{mc^2}{h^2}+\text{constant terms of order } c^{-2}+\alpha\frac{mc^2}{h^2}\epsilon\cos(\phi-\phi_0)$$
$$\times\left((4\alpha^2+4\alpha\gamma-2\beta)\frac{m^2c^2}{h^2}-\frac{16\Delta\alpha m^2a^2c^2\Omega\cos\lambda}{5h^3}\right)+O(c^{-4}). \tag{12}$$

Since a particular integral of

$$\frac{d^2u}{d\phi^2}+u = A\cos\phi$$

is

$$u = \tfrac{1}{2}A\phi\sin\phi,$$

equation (12) gives

$$u = \frac{\alpha mc^2}{h^2}\left[1+\epsilon\cos(\phi-\phi_0)+\left((2\alpha^2+2\alpha\gamma-\beta)\frac{m^2c^2}{h^2}-\frac{8\Delta\alpha m^2a^2c^2\Omega\cos\lambda}{5h^3}\right)\right.$$
$$\left.\times\,\epsilon(\phi-\phi_0)\sin(\phi-\phi_0)\right]$$
$$+\text{constant and periodic terms of order } c^{-2}+O(c^{-4}).$$

From this it follows that the angular advance of the perihelion, per revolution, is

$$[2\alpha(\alpha+\gamma)-\beta]2\pi\frac{m^2c^2}{h^2}-\frac{16\pi\alpha\Delta m^2a^2c^2\Omega\cos\lambda}{5h^3}. \tag{13}$$

The first term of equation (13) is the usual one for a non-rotating body (compare with, for example, equation (25) of I with $m_2 = 0$), and the second term is the addition caused by the rotation of the central body.

In general relativity $\alpha = \Delta = 1$, and in this case the second term of equation (13) agrees with the result given by Lense and Thirring (1918). (Actually, it only agrees with their result in the case $\lambda = 0$, but agrees with the correction to their result given by

Kalitzin (1958) for the case $\lambda \neq 0$). In both the Brans–Dicke and Nordtvedt scalar–tensor theories $\alpha = 1$ and $\Delta = (3+2\omega)/(4+2\omega)$, where ω is the dimensionless constant of the theories. So in these two theories the general relativistic result is multiplied by the factor $(3+2\omega)/(4+2\omega)$.

4. Precession of normal of orbit

The rate at which the normal to the orbit of the test particle precesses about the axis of rotation of the central body can be found from equation (5).

Since

$$r^2\dot{\phi} = h + O(c^{-2})$$

it follows that

$$\ddot{z} = \frac{h^2}{r^3}\left[\frac{d^2}{d\phi^2}\left(\frac{z}{r}\right) - z\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)\right] + O(c^{-4}).$$

But from equation (9)

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = \alpha\frac{mc^2}{h^2} - \frac{1}{r} + O(c^{-2}),$$

and so

$$\ddot{z} = \frac{h^2}{r^3}\left[\frac{d^2}{d\phi^2}\left(\frac{z}{r}\right) + \frac{z}{r} - \alpha\frac{mc^2}{h^2}z\right] + O(c^{-4}).$$

Hence equation (5) can be written as

$$\frac{d^2}{d\phi^2}\left(\frac{z}{r}\right) + \frac{z}{r} = -\frac{4\Delta ma^2\Omega\sin\lambda}{5h^2}\left(\dot{x} - 3(\boldsymbol{r}\cdot\boldsymbol{v})\frac{x}{r^2}\right) + O(c^{-4}). \tag{14}$$

Using $x = r\cos\phi$ and equation (11), equation (14) becomes

$$\frac{d^2}{d\phi^2}\left(\frac{z}{r}\right) + \frac{z}{r} = \frac{4\alpha\Delta m^2a^2c^2\Omega\sin\lambda}{5h^3} \times [\sin\phi + 2\epsilon\sin(\phi-\phi_0)\cos\phi + \epsilon\cos(\phi-\phi_0)\sin\phi] + O(c^{-4}). \tag{15}$$

Noting that a particular integral of

$$\frac{d^2w}{d\phi^2} + w = A\sin\phi$$

is

$$w = -\tfrac{1}{2}A\phi\cos\phi,$$

it follows from equation (15) that

$$\frac{z}{r} = -\frac{1}{2}\frac{4\alpha\Delta m^2a^2c^2\Omega\sin\lambda}{5h^3}\phi\cos\phi + \text{periodic terms of order } c^{-2} + O(c^{-4}). \tag{16}$$

From the way in which the coordinate system was chosen initially (see figure 1), it can be seen from equation (16) that the value of z/r where the orbit intersects the (x, z) plane decreases by an amount $\delta\lambda$ for each revolution of the test particle (see figure 2), where

$$\delta\lambda = \frac{4\pi\alpha\Delta m^2 a^2 c^2 \Omega \sin\lambda}{5h^3}. \tag{17}$$

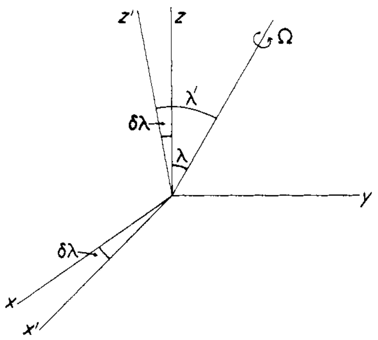

Figure 2. Coordinates for describing the motion of the normal of the orbit of the particle.

Hence the plane of the orbit appears to rotate about the y axis. But after one revolution let λ' be the inclination of the axis of rotation to the new z axis, then

$$\cos\lambda' = \cos\lambda\cos\delta\lambda$$
$$= \cos\lambda + O(c^{-4}),$$

by equation (17). So the z axis (ie the normal to the orbit) remains inclined at a constant angle λ to the axis of rotation to order c^{-2}. This means that the non-periodic effect of equation (16) will appear as a precession of the normal to the plane of the orbit about the axis of rotation. The angular precession $\delta\mu$, per revolution and in the sense of Ω, is given by

$$\delta\mu = \frac{\delta\lambda}{\sin\lambda} = +\frac{4\pi\alpha\Delta m^2 a^2 c^2 \Omega}{5h^3}. \tag{18}$$

On substituting the values $\alpha = \Delta = 1$ in equation (18) the result for general relativity is obtained, namely,

$$\delta\mu = \frac{4\pi m^2 a^2 c^2 \Omega}{5h^3}.$$

This is half the value given by Lense and Thirring (1918), which indicates that there must have been some error in their calculations. This view is supported by the fact that Kalitzin (1958) has already corrected a more substantial and obvious error in their calculation of the additional perihelion advance.

As before, in both the Brans–Dicke and Nordtvedt theories the general relativistic result is multiplied by the factor $(3+2\omega)/(4+2\omega)$.

References

Breen B 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 150–60
Kalitzin N St 1958 *Nuovo Cim.* **9** 365–74
Lense J and Thirring H 1918 *Phys. Z.* **19** 156–63